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A THEOREM ON ZETA FUNCTIONS ASSOCIATED WITH POLYNOMIALS

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ABSTRACT. Let $\beta = (\beta_1, \dots, \beta_r)$ be an r-tuple of non-negative integers and $P_j(X)$ $(j = 1, 2, \dots, n)$ be polynomials in $\mathbb{R}[X_1, \dots, X_r]$ such that $P_j(n) > 0$ for all $n \in \mathbb{N}^r$ and the series

$$\sum_{n \in \mathbb{N}^r} P_j(n)^{-s}$$

is absolutely convergent for Re $s > \sigma_j > 0$. We consider the zeta functions

$$Z(P_j, \beta, s) = \sum_{n \in \mathbb{N}^r} n^\beta P_j(n)^{-s}, \quad \text{Re } s > |\beta| + \sigma_j, \quad 1 \leq j \leq n.$$

All these zeta functions $Z(\prod_{j=1}^n P_j, \beta, s)$ and $Z(P_j, \beta, s)$ $\quad (j = 1, 2, \dots, n)$ are analytic functions of s when Re s is sufficiently large and they have meromorphic analytic continuations in the whole complex plane.

In this paper we shall prove that

$$Z(\prod_{j=1}^n P_j, \beta, 0) = \frac{1}{n} \sum_{j=1}^n Z(P_j, \beta, 0).$$

As an immediate application, we use it to evaluate the special values of zeta functions associated with products of linear forms as considered by Shintani and the first author.

1. Introduction and notation

As usual, we let $\mathbb{R}$ and $\mathbb{C}$ be the field of real numbers and the field of complex numbers, respectively. $\mathbb{N}$ denotes the set of positive integers.

Let $P_j(X_1, \dots, X_r)$ $(j = 1, \dots, n)$ be polynomials of r variables such that $P_j(n) > 0$ for all $n \in \mathbb{N}^r$ and the series

$$\sum_{n \in \mathbb{N}^r} P_j(n)^{-s} = \sum_{n_1=1}^{\infty} \dots \sum_{n_r=1}^{\infty} P(n_1, \dots, n_r)^{-s}$$

is absolutely convergent for Re $s > \sigma_j > 0$. Let $\beta = (\beta_1, \dots, \beta_r)$ be an r-tuple of non-negative integers. Consider the zeta function $Z(P_j, \beta, s)$ associated with P_j

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defined as

$$\begin{aligned} Z(P_j, \beta, s) &= \sum_{n \in \mathbb{N}^r} n^\beta P_j(n)^{-s} \\ &= \sum_{n_1=1}^{\infty} \dots \sum_{n_r=1}^{\infty} n_1^{\beta_1} \dots n_r^{\beta_r} P_j(n_1, \dots, n_r)^{-s}, \quad \operatorname{Re} s > \sigma_j + |\beta|, \end{aligned}$$

where $|\beta| = \beta_1 + \ldots + \beta_r$.

To see the analytic continuations of this kind of zeta functions, we need

Proposition 1 (Zagier [11]). *Suppose that $\varphi(s) = \sum_{\lambda>0} \alpha_\lambda \lambda^{-s}$ (λ ranges over a sequence of positive numbers tending $+\infty$) is a Dirichlet series converging for sufficiently large Re s, then $f(t) = \sum_{\lambda>0} a_\lambda e^{-\lambda t}$ is the corresponding exponential series. If at $t = 0$, $f(t)$ has the asymptotic expansion $\sum_{n \geq n_0} C_n t^{n/p}$ where n_0 is an integer and p is a fixed positive integer. Then*

1. *$\varphi(s)$ has a meromorphic continuation in the whole complex plane.*
2. *$\varphi(s)$ has a possible simple pole at $s = -n/p$, where n is not a multiple of p if $n > 0$, with residue $C_n/\Gamma(-n/p)$ and no other poles.*
3. *$\varphi(0) = C_0$.*

For a given polynomial $P(X) \in \mathbb{R}[X_1, \dots, X_r]$ such that $P(n) > 0$ for all $n \in \mathbb{N}^r$, the series

$$\sum_{n \in \mathbb{N}^r} P_j(n)^{-s}$$

is absolutely convergent for $\operatorname{Re} s > \sigma$. Suppose that

$$P(X) = \sum_{|\alpha|=0}^{p} C_\alpha X^\alpha, \qquad p = \deg P(X);$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_r$ is the degree of $X^\alpha = X_1^{\alpha_1} \cdots X_r^{\alpha_r}$. Let

$$Q(X, t) = \sum_{|\alpha|=0}^{p} C_\alpha X^\alpha t^{p-|\alpha|}.$$

Then $Q(X, t)$ is a homogeneous polynomial of degree p such that

$$Q(Xt, t) = t^p Q(X, 1) = t^p P(X).$$

As in [11], by applying the classical Euler–Maclaurin summation formula repeatedly to the function

$$\begin{aligned} g(t) &= \sum_{n \in \mathbb{N}^r} (nt)^\beta e^{-Q(nt,t)} \\ &= \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} (n_1 t)^{\beta_1} \dots (n_r t)^{\beta_r} e^{-Q(n_1 t, \dots, n_r t, t)}, \qquad t > 0, \end{aligned}$$

we get that $g(t)$ has an asymptotic expansion of the form

$$\sum_{n \geq m_0} a_n t^n$$

as t approaches 0. It follows that

$$f(t) = \sum_{n \in \mathbb{N}^r} n^\beta \exp\{-P(n)t\}$$

has an asymptotic expansion of the form

$$\sum_{n \geq n_0 - |\beta|} b_n t^{n/p}$$

as t approaches 0. By Proposition 1, $Z(P, \beta, s)$ has its meromorphic continuation in the whole complex s–plane. Therefore both the zeta functions $Z(\prod_{j=1}^n P_j, \beta, s)$ and $Z(P_j, \beta, s)$ $(j = 1, \ldots, n)$ are analytic functions of s in some half planes and have meromorphic analytic continuations in the whole complex plane.

Here we shall prove a relation concerning the special values at $s = 0$ of these zeta functions associated with polynomials.

Theorem. *Suppose that P_j $(j = 1, \ldots, n)$ are polynomials of r variables with non-negative coefficients such that $P(n) > 0$ for all $n \in \mathbb{N}^r$ and the series*

$$Z(P_j, \beta; s) = \sum_{n \in \mathbb{N}^r} n^\beta P_j(n)^{-s}$$

is absolutely convergent for Re $s > \sigma_j > 0$. Then $Z(P_j, \beta; s)$ $(j = 1, \ldots, n)$ as well as $Z(\prod_{j=1}^n P_j, \beta; s)$ have their analytic continuations. Furthermore, one has

$$Z(\prod_{j=1}^n P_j, \beta; 0) = \frac{1}{n} \sum_{j=1}^n Z(P_j, \beta; 0).$$

Let P be a polynomial having the same properties as P_j mentioned above. From the relation

$$Z(P, \beta, s - m) = \sum_{n \in \mathbb{N}^r} n^\beta P^m(n) P(n)^{-s}, \quad \operatorname{Re} s > m + \sigma + |\beta|,$$

we can express the special value $Z(P, \beta, -m)$ in terms of a linear combination of $Z(P, \alpha, 0)$ for various α. Indeed if

$$X^\beta P^m(X) = \sum_{|\alpha| = |\beta|}^{mp + |\beta|} C_\alpha X^\alpha, \ p = \deg P(X),$$

then

$$Z(P, \beta, -m) = \sum_{|\alpha| = |\beta|}^{mp + |\beta|} C_\alpha Z(P, \alpha, 0).$$

Thus, such a relation is useful to compute the special values at non-positive integers of zeta functions associated with polynomials. As applications, we shall give the formula for special values of $Z(\prod_{j=1}^n L_j, \beta, s)$ with L_j a linear form

$$L_j(X) = a_{j1} X_1 + \ldots + a_{jr} X_r + \delta_j, \quad \operatorname{Re} a_{ji} > 0,$$

as considered by the first author in [6], and the zeta function $\zeta(A, x, s)$ considered by Shintani [9], defined by

$$\zeta(A, x, s) = \sum_{n_1=0}^\infty \ldots \sum_{n_r=0}^\infty \prod_{j=1}^n (a_{j1}(n_1 + x_1) + \ldots + a_{jr}(n_r + x_r))^{-s}.$$

2. The proof of the theorem

By Proposition 1, computing $Z(P,\beta,0)$ is equivalent to computing the constant term in the asymptotic expansion at $t=0$ of the function

$$f(P,t)=\sum_{n\in\mathbb{N}^r} n^\beta \exp\{-P(n)t\}.$$

To prove our theorem, we have to consider $f(\prod_{j=1}^n P_j,t)$ and $f(P_j,t)$ $(j=1,2,\dots,n)$ simultaneously. First we need the following lemmas concerning integrations over the standard simplex of $\mathbb{R}^n$.

Let E^n be the standard simplex in $\mathbb{R}^n$ defined by

$$E^n=\big\{u=(u_1,\dots,u_n)\in\mathbb{R}^n|0\le u_i\le 1,\sum_{i=1}^n u_i=1\big\}.$$

$dU=du_1\cdots du_{n-1}$ is the standard Euclidean measure on E^n. Denote by $V_1=(1,0,\dots,0)$, $V_2=(0,1,0,\dots,0)$, $\dots$, $V_n=(0,\dots,0,1)$ the vertexes of E_n. For any positive number h, $0<h<1$, we let

$$E_j^n(h)=\big\{u=(u_1,\dots,u_n)\in E^n|1-h\le u_j\le 1\big\}.$$

Lemma 1. *Let g be a continuous function on $[0,\infty)$ and let $\alpha_1,\cdots,\alpha_{n-1}$ be non-negative integers. Then for $Re\, s>0$,*

$$\int_{E_n^n(h)} u_1^{s+\alpha_1-1}\cdots u_{n-1}^{s+\alpha_{n-1}-1} g(u_1+\cdots+u_{n-1})\,du_1\cdots du_{n-1}$$
$$=\frac{\Gamma(s+\alpha_1)\cdots\Gamma(s+\alpha_{n-1})}{\Gamma((n-1)s+\alpha_1+\cdots+\alpha_{n-1})}\int_0^h t^{(n-1)s+\alpha_1+\cdots+\alpha_{n-1}-1}g(t)\,dt.$$

Proof. With the change of variables

$$\begin{cases} t=u_1+\cdots+u_{n-1},\\ v_j=\dfrac{u_j}{t}, \qquad 1\le j\le n-1.\end{cases}$$

Then $v=(v_1,\cdots,v_{n-1})$ lies in the standard simplex E^{n-1} and the integral is transformed into

$$\int_{E^{n-1}} v_1^{s+\alpha_1-1}\cdots v_{n-1}^{s+\alpha_{n-1}-1}\,dV\int_0^h t^{(n-1)s+|\alpha|-1}g(t)\,dt.$$

Our assertion then follows from a theorem concerning β–functions of several variables [8]. □

Lemma 2. *Suppose that $f(u)$ is an analytic function on the simplex E^n and*

$$G(s)=\frac{1}{\Gamma(s)^{n-1}}\int_{E^n}(u_1\dots u_n)^{s-1}f(u)dU,\quad Re\, s>0.$$

Then $G(s)$ has its analytic continuation for $Re\ s>-1$ and $G(0)=\sum_{j=1}^n f(V_j)$.

Proof. Consider the integral around V_j defined by

$$G_j(s)=\frac{1}{\Gamma(s)^{n-1}}\int_{E_j^n(h)}(u_1\dots u_n)^{s-1}f(u)dU.$$

Suppose that on $E_j^n(h)$, f has the expansion

$$f(u) = f(V_j) + \sum_{|\alpha|\geq 1} C_\alpha u_1^{\alpha_1} \cdots u_{j-1}^{\alpha_{j-1}} u_{j+1}^{\alpha_{j+1}} \cdots u_n^{\alpha_n}.$$

Now consider the case $j = n$. From Lemma 1 and a term by term integration, for $\operatorname{Re} s > 0$,

$$\begin{aligned} G_n(s) &= f(V_n) \frac{1}{\Gamma((n-1)s)} \int_0^h t^{(n-1)s-1}(1-t)^{s-1}\, dt \\ &\quad + \sum_{|\alpha|\geq 1} C_\alpha \frac{\Gamma(s+\alpha_1)\cdots\Gamma(s+\alpha_{n-1})}{\Gamma(s)^{n-1}\Gamma((n-1)s+|\alpha|)} \int_0^h t^{(n-1)s+|\alpha|-1}(1-t)^{s-1}\, dt \end{aligned}$$

In the above expression for $G_n(s)$, write the first term as

$$\begin{aligned} &f(V_n)\frac{1}{\Gamma((n-1)s)}\left(\int_0^h t^{(n-1)s-1}\, dt + \int_0^h t^{(n-1)s-1}\big[(1-t)^{s-1}-1\big]\, dt\right) \\ &= f(V_n)\frac{h^{(n-1)s}}{\Gamma((n-1)s+1)} + f(V_n)\frac{1}{\Gamma((n-1)s)} \int_0^h t^{(n-1)s-1}\big[(1-t)^{s-1}-1\big]\, dt. \end{aligned}$$

Hence $G_n(s)$ has analytic continuation for $\operatorname{Re} s > -1$ and its special value at $s = 0$ is given by $f(V_n)$.

For the remaining terms, the integrals

$$\int_0^h t^{(n-1)s+|\alpha|-1}(1-t)^{s-1}\, dt, \qquad |\alpha| \geq 1,$$

are convergent for $s = 0$ and the gamma functions

$$\frac{\Gamma(s+\alpha_1)\cdots\Gamma(s+\alpha_{n-1})}{\Gamma(s)^{n-1}\Gamma((n-1)s+|\alpha|)}, \quad |\alpha| \geq 1,$$

are analytic at $s = 0$ and vanish there. Thus we have

$$G_n(0) = f(V_n).$$

In general, we have

$$G_j(0) = f(V_j).$$

Note that $H(s) = G(s) - \sum_{j=1}^n G_j(s)$ vanishes at $s = 0$ owing to the zero of order $n-1$ from the factor $\Gamma(s)^{1-n}$. Thus

$$G(0) = \sum_{j=1}^n G_j(0) = \sum_{j=1}^n f(V_j). \qquad \square$$

Proof of the theorem. The existence of analytic continuations follows from a standard argument as mentioned before (see also [3]). Here we prove the equality. For $\operatorname{Re} s > \max(\sigma_1, \ldots, \sigma_n)$, we have

$$\begin{aligned} \prod_{j=1}^n P_j(n)^{-s}[\Gamma(s)]^n &= \int_0^\infty \cdots \int_0^\infty (t_1 \cdots t_n)^{s-1} \exp\{-\sum_{j=1}^n P_j(n) t_j\} dt_1 \cdots dt_n \\ &= \int_0^\infty t^{ns-1} dt \int_{E^n} (u_1 \cdots u_n)^{s-1} \exp\{-\sum_{j=1}^n P_j(n) u_j t\} dU. \end{aligned}$$

It follows for sufficiently large Re s,

$$Z(\prod_{j=1}^{n} P_j, \beta; s)\Gamma(s) = \int_0^\infty t^{ns-1} dt \frac{1}{\Gamma(s)^{n-1}} \int_{E^n} (u_1 \cdots u_n)^{s-1} f(u,t) dU$$

where $f(u,t) = \sum_{n\in\mathbb{N}^r} n^\beta \exp\{-\sum_{j=1}^n P_j(n) u_j t\}$. Set

$$F(s,t) = \frac{1}{\Gamma(s)^{n-1}} \int_{E^n} (u_1 \ldots u_n)^{s-1} f(u,t) dU.$$

As a function of s, $F(s,t)$ is an analytic function of s for $\operatorname{Re} s > 0$. By the previous lemma, $F(s,t)$ has its analytic continuation for $\operatorname{Re} s > -1$. Furthermore,

$$F(0,t) = \sum_{j=1}^{n} \sum_{n\in\mathbb{N}^r} n^\beta \exp\{-P_j(n)t\}.$$

For any $\epsilon > 0$, let $L(\epsilon)$ be the contour in a complex plane consisting of $[\epsilon, +\infty)$ twice in the opposite direction and the circle $|t| = \epsilon$ counterclockwise. Then we have

$$\int_{L(\epsilon)} t^{ns-1} F(s,t)\, dt = (e^{2\pi ins} - 1) \int_\epsilon^\infty t^{ns-1} F(s,t)\, dt + \int_{|t|=\epsilon} t^{ns-1} F(s,t)\, dt.$$

For sufficiently large $\operatorname{Re} s$, the second integral approaches to zero as $\epsilon \to 0$. Consequently for sufficiently small $\epsilon > 0$ and sufficiently large $\operatorname{Re} s$, one has

$$Z(\prod_{j=1}^{n} P_j, \beta; s)\Gamma(s) = \frac{1}{e^{2\pi ins} - 1} \int_{L(\epsilon)} t^{ns-1} F(s,t)\, dt.$$

With the well–known functional equation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = \frac{2\pi i e^{\pi i s}}{e^{2\pi i s} - 1},$$

we rewrite the formula as

$$Z(\prod_{j=1}^{n} P_j, \beta; s) = \Gamma(1-s) e^{-\pi i s} \frac{e^{2\pi i s} - 1}{e^{2\pi ins} - 1} \cdot \frac{1}{2\pi i} \int_{L(\epsilon)} t^{ns-1} F(s,t)\, dt.$$

Note that the above contour integral is absolutely convergent for all s, hence it gives the analytic continuation of $Z(\prod_{j=1}^n P_j, \beta; s)$. When $s = 0$, the integration along $[\epsilon, +\infty)$ twice in the opposite direction will be cancelled. Hence

$$\begin{aligned} Z(\prod_{j=1}^{n} P_j, \beta; 0) &= \frac{1}{n} \cdot \frac{1}{2\pi i} \int_{|t|=\epsilon} t^{-1} F(0,t)\, dt \\ &= \frac{1}{n} \times \{\text{the constant term in the} \\ &\qquad\qquad \text{asymptotic expansion of } F(0,t) \text{ at } t = 0\}. \end{aligned}$$

On the other hand, we have for Re $s > \sigma_j + |\beta|$,

$$Z(P_j, \beta; s)\Gamma(s) = \int_0^\infty t^{s-1} \sum_{n\in\mathbb{N}^r} n^\beta \exp\{-P_j(n)t\} dt.$$

Hence

$$Z(P_j, \beta; 0) = \text{the constant term in the asymptotic expansion of} \quad \sum_{n\in\mathbb{N}^r} n^\beta \exp\{-P_j(n)t\} \quad \text{at} \quad t = 0.$$

Thus our assertion follows. □

Remark. The condition that $P_j(X)$ $(j = 1, \dots, n)$ are polynomials of non-negative coefficients such that $P(n) > 0$ for all $n \in \mathbb{N}^r$ can be relaxed. Indeed, Re $P_j(n) > 0$ $(j = 1, \dots, n)$, along with some convergence condition on the series, is enough for our purpose.

3. Applications

Here we only mention two main applications of our theorem. Other applications to evaluate the special values of particular zeta functions associated with polynomials are also possible.

Let $P(X)$ be a product of n linear forms in r variables $X_1, \dots, X_r$ as given by

$$P(X_1, \dots, X_r) = \prod_{j=1}^n L_j(X) = \prod_{j=1}^n (a_{j1}X_1 + \dots + a_{jr}X_r + \delta_j)$$

$$\operatorname{Re} a_{ji} > 0, \quad \operatorname{Re}(\delta_j + \sum_{i=1}^r a_{ji}) > 0.$$

In [6], the first author considered the zeta function

$$Z(P, \beta, s) = \sum_{n_1=1}^\infty \dots \sum_{n_r=1}^\infty n_1^{\beta_1} \dots n_r^{\beta_r} P(n_1, \dots, n_r)^{-s}, \quad \operatorname{Re} s > \frac{r + |\beta|}{n},$$

and he obtained an explicit expression of $Z(P, \beta, -m)$ in terms of polynomials in Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^\infty \frac{B_n t^n}{n!}, \qquad |t| < 2\pi.$$

As mentioned before, $Z(P, \beta, -m)$ is a linear combination of $Z(P, \alpha, 0)$ for various α. Also by our theorem, $Z(P, \alpha, 0)$ is the average of $Z(L_j, \alpha, 0)$, $j = 1, \dots, n$. So it suffices to evaluate $Z(L_j, \alpha, 0)$ in order to compute $Z(P, \beta, -m)$.

Here we use the notation from [6] and let J^p be the linear extension of $\mathbb{C}[X_1, \dots, X_p]$ to $\mathbb{C}$ satisfying

$$J^p[X_1^{\alpha_1} \dots X_p^{\alpha_p}] = \prod_{i=1}^p \zeta(-\alpha_i) = \prod_{i=1}^p \frac{(-1)^{\alpha_i} B_{\alpha_i+1}}{\alpha_i + 1}.$$

Proposition 2. *Let $L(X) = a_1x_1 + \dots + a_rx_r + \delta$ be a linear form with $Re\, a_i > 0$, $Re\,(\delta + \sum_{i=1}^r a_i) > 0$. Then the zeta function $Z(L, \alpha, s)$ defined by*

$$Z(L, \alpha, s) = \sum_{n_1=1}^\infty \dots \sum_{n_r=1}^\infty n_1^{\alpha_1} \dots n_r^{\alpha_r} (a_1n_1 + \dots + a_rn_r + \delta)^{-s}, \quad Re\, s > r + |\alpha|,$$

has an analytic continuation in the whole complex plane, which is analytic except for simple poles at $s = 1, 2, \dots, r + |\alpha|$. *Furthermore*

$$Z(L, \alpha, 0) = \sum_{\{j_1,\dots,j_p\}} J^{r-p}[\int_{\Delta(X_{j_1},\dots,X_{j_p})} X^\alpha dX_{j_1} \dots dX_{j_p}]$$

where $\{j_1, \dots, j_p\}$ *ranges over all subsets of* $\{1, 2, \dots, r\}$ *in the summation and* $\Delta(X_{j_1}, \dots, X_{j_p})$ *is the domain in* $\mathbb{R}^p$ *defined by* $L(X) \geq 0, X_{j_1} \leq 0, \dots, X_{j_p} \leq 0$ *when the coefficients* a_i $(i = 1, \dots, r)$ *and* δ *are real numbers.*

Proof. For Re $s > 0$, we have

$$(a_1n_1 + \dots + a_rn_r + \delta)^{-s}\Gamma(s) = \int_0^\infty t^{s-1} \exp\{-(a_1n_1 + \dots + a_rn_r + \delta)t\}dt.$$

Hence for Re $s > r + |\alpha|$, we have

$$Z(L, \alpha, s)\Gamma(s) = \sum_{n\in\mathbb{N}^r} n^\alpha \int_0^\infty t^{s-1} \exp\{-(a_1n_1 + \dots + a_rn_r + \delta)t\}dt.$$

$$= \int_0^\infty t^{s-1}e^{-\delta t} \prod_{i=1}^r (\sum_{n=1}^\infty n^{\alpha_i} e^{-a_int})dt.$$

It follows from Proposition 1 that

$$Z(L, \alpha, 0) = \text{ the constant term in the expansion at } t = 0 \text{ of the function } F(t) = e^{-\delta t} \prod_{i=1}^r (\sum_{n=1}^\infty n^{\alpha_i} e^{-a_int}).$$

Note that

$$\sum_{n=1}^\infty n^{\alpha_i} e^{-a_int} = \frac{\alpha_i!}{(a_it)^{\alpha_i+1}} + \sum_{n\geq\alpha_i+1} \frac{(-1)^{\alpha_i}B_n(a_it)^{n-\alpha_i-1}}{n \cdot (n - \alpha_i - 1)!} \quad (i = 1, \dots, r).$$

Thus the constant in $F(t)$ is a sum of products of the form

$$\prod_{i=1}^p \frac{\alpha_i!}{(a_i)^{\alpha_i+1}} \sum_{|\beta|=\alpha_1+\dots+\alpha_p+p} \prod_{i=p+1}^r \frac{(-1)^{\alpha_i}B_{\beta_i+\alpha_i+1}a_i^{\beta_i}}{(\beta_i + \alpha_i + 1) \cdot \beta_i!} \cdot \frac{(-1)^{\beta_{r+1}}\delta^{\beta_{r+1}}}{\beta_{r+1}!}$$

after a permutation in the index set $\{1, 2, \dots, r\}$. Here $|\beta| = \beta_{p+1} + \dots + \beta_r + \beta_{r+1}$.

On the other hand, an elementary calculation shows that

$$\int_{\Delta(X_1,\dots,X_p)} X^\alpha dX_1 \dots dX_p$$

$$= \left[\prod_{i=1}^p \frac{\alpha_i!}{(a_i)^{\alpha_i+1}}\right] \frac{\prod_{i=p+1}^r X_i^{\alpha_i}[-(a_{p+1}X_{p+1} + \dots + a_rX_r + \delta)]^{\alpha_1+\dots+\alpha_p+p}}{(\alpha_1 + \dots + \alpha_p + p)!}.$$

Thus the one-to-one correspondence between the constant terms in the product $F(t)$ and polynomials arising from integrations of X^α over various simplexes is clear. Hence our assertion follows. □

As an immediate consequence of our theorem and Proposition 2, we have the following formula for the special values of $Z(P, \beta, s) = Z(\prod_{j=1}^n L_j, \beta, s)$.

Corollary 1 (Eie [6]). *The special value of $Z(P,\beta,s)$ at nonpositive integers $-m$ is given by*

$$Z(P,\beta,-m)=\frac{1}{n}\sum_{j=1}^{n}\sum_{\{j_1,\dots,j_p\}}J^{r-p}[\int_{\Delta_j(X_{j_1},\dots,X_{j_p})}X^{\beta}P^m(X)dX_{j_1}\dots dX_{j_p}].$$

Here in the second summation, $\{j_1,\dots,j_p\}$ ranges over all subsets of $\{1,\dots,r\}$. The domain of integration $\Delta_j(X_{j_1},\dots,X_{j_p})$ is defined by $(X_{j_1},\dots,X_{j_p})\in\mathbb{R}^p$, $L_j(X)\geq 0$, $X_{j_1}\leq 0,\dots,\dots,X_{j_p}\leq 0$, when the coefficients of $L_j(X)$ are positive real numbers.

The exact same procedure can be used to evaluate the special values of the zeta function $\zeta(A,x,s)$ considered by Shintani in [9] in order to determine the special values of Dedekind zeta functions of totally real number fields. Shintani's formulas for the special values of $\zeta(A,x,s)$ were improved by the first author in the present form below [7]. For any $r\times n$ matrix $A=[a_{ji}]$ with positive entries and any r-tuple of complex numbers $x=(x_1,\dots,x_r)$. The zeta function $\zeta(A,x,s)$ is defined as

$$\zeta(A,x,s)=\sum_{n_1=0}^{\infty}\dots\sum_{n_r=0}^{\infty}\prod_{j=1}^{n}[a_{j1}(n_1+x_1)+\dots+a_{jr}(n_r+x_r)]^{-s},\quad \operatorname{Re}s>\frac{r}{n}.$$

Let $B_n(x)$ be the Bernoulli polynomial defined by $B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k$.

For any subset S of $I=\{1,2,\dots,r\}$ and polynomial $g(u)$ with variables in $\{u_i\mid i\in S\}$ and $g(u)=\sum_{|\alpha|=0}^{p}b_\alpha\prod_{i\in S}u_i^{\alpha_i}$, we let

$$J_S[g]=\sum_{|\alpha|=0}^{p}b_\alpha\cdot\prod_{i\in S}\frac{-B_{\alpha_i+1}(x_i)}{\alpha_i+1}$$

if $g(u)=\sum_{|\alpha|=0}^{p}b_\alpha\prod_{i\in S}u_i^{\alpha_i}$. When $S=\phi$, we let $J_\phi[c]=c$ for any constant c.

Then Shintani's formula for $\zeta(A,x,1-m)$ can be improved by the following.

Corollary 2 (Shintani [8]). *For any positive integer m,*

$$\zeta(A,x,1-m)=\frac{1}{n}\sum_{S}J_S\left[\sum_{j=1}^{n}\int_{\Delta_{j,S}(u)}P^{m-1}(u)\prod_{j\notin S}du_j\right].$$

Here S ranges over all non-empty subsets of $I=\{1,2,\dots,r\}$,

$$P(u)=\prod_{j=1}^{n}[a_{j1}u_1+\dots+a_{jr}u_r]$$

and $\Delta_{j,S}(u)$ is the simplex defined by $\Delta_{j,S}(u):(u_j)_{j\notin S},\ u_j\leq 0,\ L_j(u)=a_{j1}u_1+\dots+a_{jr}u_r\geq 0$.

4. A further identity

Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and n be a positive integer. Then the following identity is well known ([1], page 276):

$$\begin{aligned}
&2^{2n}\sum_{k=0}^{n+1}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\frac{B_{2k}}{(2k)!}\alpha^{n+1-k}(-\beta)^k \\
&= -\alpha^{-n}\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\} + (-\beta)^{-n}\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\}.
\end{aligned}$$

Here we extend it to an identity of Bernoulli polynomials.

Proposition 3. *For positive numbers α, β with $\alpha\beta = \pi^2$ and real numbers u, v such that $0 \le u, v \le 1$, we have*

$$\begin{aligned}
\text{(a)}\qquad &2^{2n}\sum_{k=0}^{n+1}\frac{B_{2k}(v)}{(2k)!}\frac{B_{2n-2k+2}(u)}{(2n-2k+2)!}\alpha^{n-k+1}(-\beta)^k \\
&= -\frac{1}{2}\alpha^{-n}\sum_{k=1}^{\infty}\frac{k^{-2n-1}\cos(2k\pi v)[e^{2ku\alpha}+e^{2k(1-u)\alpha}]}{e^{2k\alpha}-1} \\
&\qquad + \frac{1}{2}(-\beta)^{-n}\sum_{k=1}^{\infty}\frac{k^{-2n-1}\cos(2ku\pi)[e^{2k(1-v)\beta}+e^{2kv\beta}]}{e^{2k\beta}-1};
\end{aligned}$$

$$\begin{aligned}
\text{(b)}\qquad &2^{2n}\sum_{k=0}^{n}\frac{B_{2k+1}(v)}{(2k+1)!}\frac{B_{2n-2k+1}(u)}{(2n-2k+1)!}\alpha^{n-k+1/2}\beta^{k+1/2}(-1)^k \\
&= -\frac{1}{2}\alpha^{-n}\sum_{k=1}^{\infty}\frac{k^{-2n-1}\sin(2k\pi v)[e^{2ku\alpha}-e^{2k(1-u)\alpha}]}{e^{2k\alpha}-1} \\
&\qquad + \frac{1}{2}(-\beta)^{-n}\sum_{k=1}^{\infty}\frac{k^{-2n-1}\sin(2k\pi u)[e^{2k(1-v)\beta}-e^{2kv\beta}]}{e^{2k\beta}-1}.
\end{aligned}$$

Proof. For the time being, we suppose $u, v > 0$. For any given $\epsilon > 0$, consider the zeta function

$$Z_\epsilon(s) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left[\sqrt{\alpha}(n_1+u)+(\epsilon+i\sqrt{\beta})(n_2+v)\right]^{-s}, \quad \operatorname{Re} s > 2.$$

$Z_\epsilon(s)$ has its analytic continuation and its special value at $s = -2n$ is

$$Z_\epsilon(-2n) = \frac{(2n)!}{\sqrt{\alpha}(\epsilon+i\sqrt{\beta})}\sum_{k=0}^{2n+2}\frac{B_k(v)}{k!}\frac{B_{2n-k+2}(u)}{(2n-k+2)!}(\sqrt{\alpha})^{2n-k+2}(\epsilon+i\sqrt{\beta})^k$$

by Corollary 2. On the other hand, we set

$$F_\epsilon(t) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \exp\{-[\sqrt{\alpha}(n_1+u) + (\epsilon + i\sqrt{\beta})(n_2+v)t]\}$$
$$= \frac{\exp[-(\sqrt{\alpha}u + \epsilon v + i\sqrt{\beta}v)t]}{(1-e^{-\sqrt{\alpha}t})(1-e^{-(\epsilon+i\sqrt{\beta})t})} ,$$
$$F(t) = \frac{\exp[-(\sqrt{\alpha}u + i\sqrt{\beta}v)t]}{(1-e^{-\sqrt{\alpha}t})(1-e^{-i\sqrt{\beta}t})} .$$

By Proposition 1, we have

$$Z_\epsilon(-2n) = (2n)! \times \text{the coefficient of } t^{2n} \text{ in the asymptotic expansion at } t=0 \text{ of the function } F_\epsilon(t).$$

It follows that for any $\epsilon > 0$, we have the identity

$$\frac{(2n)!}{\sqrt{\alpha}(\epsilon + i\sqrt{\beta})} \sum_{k=0}^{2n+2} \frac{B_k(v)}{k!} \frac{B_{2n-k+2}(u)}{(2n-k+2)!} (\sqrt{\alpha})^{2n-k+2} (\epsilon + i\sqrt{\beta})^k$$
$$= (2n)! \times \text{the coefficient of } t^{2n} \text{ in the asymptotic expansion at } t=0 \text{ of the function } F_\epsilon(t).$$

As ϵ approaches 0, both sides in the formula above converge to nice limits, respectively. Thus we have

$$\frac{1}{\sqrt{\alpha\beta}i} \sum_{k=0}^{2n+2} \frac{B_k(v)}{k!} \frac{B_{2n-k+2}(u)}{(2n-k+2)!} (\sqrt{\alpha})^{2n-k+2} (i\sqrt{\beta})^k$$
$$= \text{the coefficient of } t^{2n} \text{ in the asymptotic expansion at } t=0 \text{ of the function } F(t)$$
$$= \frac{1}{2\pi i} \int_{|z|=\delta} z^{-2n-1} F(z)\, dz ,$$

where δ is a positive number such that $\delta < \min(2\sqrt{\alpha}, 2\sqrt{\beta})$ and the direction on the circle $|z| = \delta$ is counterclockwise. By the residue theorem, the contour integral is equal to

$$-\sum_{k\in\mathbb{Z}, k\neq 0} \{ \text{ Residue of } z^{-2n-1}F(z) \text{ at } z = \frac{2k\pi i}{\sqrt{\alpha}},\ \frac{2k\pi}{\sqrt{\beta}} \}$$
$$= -\frac{1}{2^{2n+1}} \cdot \frac{1}{\sqrt{\alpha\beta}i} \alpha^{-n} \sum_{k=1}^{\infty} \frac{k^{-2n-1}(e^{-2k\pi iv + 2k(1-u)\alpha} + e^{2k\pi iv + 2ku\alpha})}{e^{2k\alpha}-1}$$
$$+ \frac{1}{2^{2n+1}} \cdot \frac{1}{\sqrt{\alpha\beta}i} (-\beta)^{-n} \sum_{k=1}^{\infty} \frac{k^{-2n-1}(e^{-2k\pi ui + 2k\beta v} + e^{2k\pi ui + 2k(1-v)\beta})}{e^{2k\beta}-1}.$$

Comparing the real and imaginary parts of the resulted formula, we get our assertion for $u, v > 0$.

Note that both sides of the identities in (a) and (b) are analytic functions of u and v, it holds for $u > 0$, $v > 0$, so it also holds for $u = 0$ or $v = 0$. □

Remark. When n is a negative integer, we have

$$\frac{1}{2\pi i}\int_{|z|=\delta} z^{-2n-1}F(z)\,dz = 0$$

if $n < -1$ since $z^2F(z)$ is analytic at $z = 0$. It follows that for $0 < u,\ v < 1$ and $m > 1$,

$$\alpha^m \sum_{k=1}^{\infty} \frac{k^{2m-1}\cos(2k\pi v)[e^{2ku\alpha}+e^{2k(1-u)\alpha}]}{e^{2k\alpha}-1}$$
$$= (-\beta)^m \sum_{k=1}^{\infty} \frac{k^{2m-1}\cos(2k\pi u)[e^{2ku\beta}+e^{2k(1-v)\beta}]}{e^{2k\beta}-1}\ .$$

For the limit case $u = v = 0$, we have

$$\alpha^m[\zeta(1-2m)+2\sum_{k=1}^{\infty}\frac{k^{2m-1}}{e^{2k\alpha}-1}] = (-\beta)^m[\zeta(1-2m)+2\sum_{k=1}^{\infty}\frac{k^{2m-1}}{e^{2k\beta}-1}]\ .$$

This is equivalent to

$$\alpha^m\sum_{k=1}^{\infty}\frac{k^{2m-1}}{e^{2k\alpha}-1} - (-\beta)^m\sum_{k=1}^{\infty}\frac{k^{2m-1}}{e^{2k\beta}-1} = [\alpha^m-(-\beta)^m]\frac{B_{2m}}{4m}$$

as given on page 261 in [1].

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